

Moderate deviations for stabilizing functionals in geometric probability

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Abstract

The purpose of the present paper is to establish explicit bounds on moderate deviation probabilities for a rather general class of bounded geometric functionals enjoying the exponential stabilization property. Compared to our previous work [1], the price to pay for the considerably larger generality of our estimates is a narrower scale range in which our moderate deviation results hold; we argue that a range limitation seems inevitable under our general assumptions though. Our proof techniques rely on cumulant expansions and cluster measures and yield completely explicit bounds on deviation probabilities. The examples of geometric functionals we treat include bounded statistics of random packing models and random graphs arising in computational geometry, such as Euclidean nearest neighbor graphs, Voronoi graphs and sphere of influence graphs.

Keywords: Stabilizing functionals, moderate deviations, explicit bounds, cumulants

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1. Introduction, main results

1.1. Introduction

Stabilization is an important concept expressing in natural geometric terms mixing properties of a broad class of functionals of point processes arising in geometric probability, see [21, 22, 3]. Even though these processes are presumably also tractable using more traditional mixing concepts, stabilization-based techniques proved extremely convenient in studying the asymptotic behavior of large random geometric systems. This is due to the geometric nature of these methods which makes them compatible with many stochastic geometric set-ups in which the target functionals arise. In particular, stabilization is often helpful in establishing a direct connection between the microscopic (local) and macroscopic properties of the processes studied, see *ibidem* for further details. Stabilization has been successfully used to establish laws of large numbers for many functionals [22, 23] and it has also been employed in a general setting to establish Gaussian limits for re-normalized functionals as well as re-normalized spatial point measures [3, 18]. The functionals to which the afore-mentioned theory applies include those defined by percolation models [18], random graphs in computational geometry [3, 21], random packing models [2, 22], germ grain models [3], and the process of maximal points [4]. Large deviation principle for stabilizing functionals have also been established, see [28].

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Finally, the corresponding moderate deviation principle, interpolating between the central limit theorem and law of large numbers, has been obtained in [1] and [13], for a rather limited sub-class of the above examples though, namely for empirical functionals of random sequential packing, nearest neighbor graphs and germ-grain models.

In this paper we use stabilization combined with cumulant expansion techniques in the spirit of [3] to prove moderate deviation bounds for a much richer class of general bounded exponentially stabilizing functionals, extending the above list of tractable functionals with those of Voronoi–Delaunay graphs, sphere of influence graphs as well as more general random packing and birth-growth models (lifting the unnatural lower bound on grain sizes). Moreover, our deviation probability estimates are much more explicit than those established in [1]. On the other hand though, the range of scaling regimes to which our moderate deviation results apply is much narrower than that treated in [1]. However, it is well known that many natural multidimensional stochastic systems exhibiting various types of exponential mixing often satisfy the Gaussian moderate deviation principle only up to a certain point beyond the CLT scale, whereafter the Gaussian behavior breaks down and gets replaced by phenomena of a different nature, a spectacular example being the phase separation, condensation and droplet creation as established for many statistical mechanical models in phase transition regime, see the seminal monograph [11] as well as the survey [6]. Consequently, we believe the Gaussian moderate deviation principle may well be violated by geometric stabilizing functionals for ranges far enough from the CLT regime, and even though we are definitely not in a position to claim that the ranges of Gaussian behavior for deviation probabilities established in this paper are optimal (especially that better ranges were obtained in [1] for a restricted class of models), most likely we should not hope to get a full range Gaussian moderate deviation principle at the level of generality considered here.

1.2. Terminology

Borrowing the notation of [3, 23] we let $\xi(x; \mathcal{X})$ be a measurable \mathbb{R} -valued function defined for all pairs (x, \mathcal{X}) , where $\mathcal{X} \subset \mathbb{R}^d$ is finite and where $x \in \mathcal{X}$. Assume that ξ is bounded in absolute value by some $C_\xi < \infty$ and translation invariant, that is $\xi(x; \mathcal{X}) = \xi(x - y; \mathcal{X} - y)$ for all $y \in \mathbb{R}^d$. When $x \notin \mathcal{X}$, we abbreviate notation and write $\xi(x; \mathcal{X})$ instead of $\xi(x; \mathcal{X} \cup x)$. For all $\lambda > 0$ let $\xi_\lambda(x; \mathcal{X}) := \xi(\lambda^{1/d}x; \lambda^{1/d}\mathcal{X})$, where given $a > 0$, we let $a\mathcal{X} := \{ax; x \in \mathcal{X}\}$. $\xi(x; \mathcal{X})$ should be thought of as quantifying the interaction between x and the point set \mathcal{X} .

For a locally integrable function $g: \mathbb{R}^d \rightarrow [0, \infty)$, denote by \mathcal{P}_g a Poisson point process with intensity g . We shall focus on *random point measures* on \mathbb{R}^d defined by:

$$\mu_{\lambda\kappa}^\xi := \sum_{x \in \mathcal{P}_{\lambda\kappa}} \xi_\lambda(x; \mathcal{P}_{\lambda\kappa}) \delta_x,$$

where $\lambda > 0$ and where κ is a bounded probability density on \mathbb{R}^d (notice that $\lambda\kappa$ in $\mu_{\lambda\kappa}^\xi$ denotes a double index, whereas in $\mathcal{P}_{\lambda\kappa}$, it denotes a product). More precisely, we will establish moderate deviation principle for the centered version of $\mu_{\lambda\kappa}^\xi$, namely for $\bar{\mu}_{\lambda\kappa}^\xi := \mu_{\lambda\kappa}^\xi - \mathbb{E} \mu_{\lambda\kappa}^\xi$.

Some of our applications are also based on marked Poisson point processes. Instead of considering a separate probability space for the marks, we consider ξ as a *random* geometric functional. More precisely, if Ξ is a deterministic functional defined on pairs $((x_1, t_1); \{(x_1, t_1), \dots, (x_n, t_n)\})$, where $x_i \in \mathbb{R}^d$ as before, while t_i belong to a so called *mark space* \mathcal{M} , we allow ξ to be defined as:

$$\xi(x_1; \{x_1, \dots, x_n\}) := \Xi((x_1, T(x_1)); \{(x_1, T(x_1)), \dots, (x_n, T(x_n))\}), \quad (1.1)$$

where $T(x), x \in \mathbb{R}^d$, are random *marks*, which are independent and identically distributed.

For marked point processes, translation invariance will be considered as follows: the functional Ξ will be called translation invariant if $\Xi((x_1, t_1); \{(x_1, t_1), \dots, (x_n, t_n)\}) = \Xi((x_1 - y, t_1); \{(x_1 - y, t_1), \dots, (x_n - y, t_n)\})$ for all $y \in \mathbb{R}^d$; the random functional ξ defined in (1.1) will be called translation invariant if arising from a translation invariant functional Ξ .

Now we turn to exponential stabilization, which, used heavily in [3, 21], plays a central role in all that follows. For $x \in \mathbb{R}^d$ and $r > 0$, $B_r(x)$ denotes the Euclidean ball centered at x of radius r , and $\mathbf{0}$ denotes the origin of \mathbb{R}^d .

Definition 1.1. A (fixed) geometric functional ξ *stabilizes* at x with respect to a locally finite set \mathcal{X} if there exists a positive real number R (*radius of stabilization*), such that:

$$\xi\left(x; (\mathcal{X} \cap B_R(x)) \cup \mathcal{Y}\right)$$

is independent of \mathcal{Y} for all finite $\mathcal{Y} \subset \mathbb{R}^d \setminus B_R(x)$. In other words, the interaction between x and a point set is unaffected by changes outside $B_R(x)$. In particular, for $r \geq R(x)$, $\xi(x; \mathcal{X} \cap B_r(x))$ does not depend on r ; this allows us to extend the definition of $\xi(x, \mathcal{X})$ to locally finite point sets in the obvious way.

Definition 1.2. Let $\rho: [0, \infty) \rightarrow [0, \infty)$ be a function with $\lim_{r \rightarrow \infty} \rho(r) = 0$. A (possibly random) geometric functional ξ stabilizes with decay rate ρ with respect to an almost surely locally finite point process \mathcal{X} if for each $x \in \mathbb{R}^d$, there exists a random variable $R(x)$, which is almost surely a radius of stabilization of ξ at x with respect to \mathcal{X} and such that $\mathbb{P}(R(x) \geq t) \leq \rho(t)$ for all $t \geq 0$.

Remark 1.1. Suppose that ξ stabilizes with a certain decay rate ρ with respect to a Poisson point process \mathcal{P}_g , where the intensity g is locally integrable. Then for each fixed x , one can almost surely define $\xi(x; \mathcal{P}_g)$ because \mathcal{P}_g is almost surely locally finite.

Definition 1.3. A deterministic geometric functional ξ is *exponentially stabilizing* for a probability density κ if there exist constants L, α and $\lambda_0 > 0$, such that for all $\lambda \geq \lambda_0$, ξ_λ stabilizes with decay rate $L \exp(-\alpha \lambda^{1/d} t)$ with respect to $\mathcal{P}_{\lambda\kappa}$.

Definition 1.4. A deterministic geometric functional ξ is *strongly exponentially stabilizing* for a probability density κ if there exist constants L, α, B and $\lambda_0 > 0$, such that for all $\lambda \geq \lambda_0$, the functional $(x, \mathcal{X}) \mapsto \xi_\lambda(x; \mathcal{X} \cup \mathcal{Y})$ stabilizes with decay rate $LB^k \exp(-\alpha \lambda^{1/d} t)$ with respect to $\mathcal{P}_{\lambda\kappa}$.

Definition 1.5. A random geometric functional ξ defined as in (1.1) is (strongly) exponentially stabilizing for a probability density κ if the corresponding definition for fixed functionals holds for Poisson point processes $\mathcal{P}_{\lambda\kappa}$ which are independent of the marks $\{T(x) ; x \in \mathbb{R}^d\}$; Definition 1.2 is taken with unconditional probability.

See [3] and our Section 2 below for examples of exponentially stabilizing functionals.

In our main results, we shall make the following assumptions on κ and ξ :

Assumptions ()*:

- κ is a probability density function on \mathbb{R}^d , which is bounded and Lebesgue-almost everywhere continuous.
- ξ is deterministic or defined as in (1.1). In the latter case, we assume that the Poisson point process $\mathcal{P}_{\lambda\kappa}$ is independent of the marks $\{T(x) ; x \in \mathbb{R}^d\}$.
- ξ is translation invariant and bounded, i. e., $|\xi| \leq C_\xi$ for some finite constant C_ξ .
- ξ is strongly exponentially stabilizing.

We let $\langle f, \mu \rangle$ denote the integral with respect to a signed finite variation Borel measure μ of a μ -integrable function f . We also write $\mathcal{B}(W)$ for the collection of bounded measurable $f: W \rightarrow \mathbb{R}$. For a set A , denote by $|A|$ its cardinality. Finally, we shall assume that $0/0 = 0$ (and $a/0 = +\infty$ for $a > 0$ and $-\infty$ for $a < 0$).

1.3. Known results

From [3, 23] we know that our boundedness and strong exponential stabilization assumptions imposed on ξ are more than enough to guarantee that the one and two point correlation functions for $\xi_\lambda(x; \mathcal{P}_{\lambda\kappa})$ converge in the large λ limit, which establishes volume order asymptotics for $\mathbb{E}[\mu_{\lambda\kappa}^\xi([0, 1]^d)]$ and $\text{Var}[\mu_{\lambda\kappa}^\xi([0, 1]^d)]$ as $\lambda \rightarrow \infty$; more generally, the corresponding asymptotics for integrals $\mathbb{E}\langle f, \mu_{\lambda\kappa}^\xi \rangle$ and $\sigma_\lambda^2[f] := \text{Var}\langle f, \mu_{\lambda\kappa}^\xi \rangle$ is also known for bounded measurable test functions. Moreover, under our assumptions on ξ , it is known that the limit of the re-normalized measures $\lambda^{-1/2} \overline{\mu}_{\lambda\kappa}^\xi$ is a generalized mean zero Gaussian field in the sense that the finite-dimensional distributions of $\lambda^{-1/2} \overline{\mu}_{\lambda\kappa}^\xi$ over $f \in \mathcal{B}(\mathbb{R}^d)$ converge to those of a Gaussian field. In formal terms, under Assumptions (*), we have the following, see Theorem 2.1 of [19] and Theorems 2.1–2.3 along with the remarks below them in [20]:

- For all $f \in \mathcal{B}(\mathbb{R}^d)$, we have:

$$\lim_{\lambda \rightarrow \infty} \frac{\mathbb{E}\langle f, \mu_{\lambda\kappa}^\xi \rangle}{\lambda} = \int_{\mathbb{R}^d} f(x) \mathbb{E}[\xi(\mathbf{0}, \mathcal{P}_{\kappa(x)})] \kappa(x) dx.$$

- For all $\tau \geq 0$, there is a constant $V^\xi(\tau)$, such that for all $f \in \mathcal{B}(\mathbb{R}^d)$, the variance $\sigma_\lambda^2[f] = \text{Var}\langle f, \mu_{\lambda\kappa}^\xi \rangle$ satisfies:

$$\lim_{\lambda \rightarrow \infty} \frac{\sigma_\lambda^2[f]}{\lambda} = Q_\kappa^\xi(f) := \int_{\mathbb{R}^d} f^2(x) V^\xi(\kappa(x)) \kappa(x) dx. \quad (1.2)$$

Moreover, the map V^ξ is measurable and bounded on bounded intervals.

- If κ has a bounded support, then the finite-dimensional distributions of $\lambda^{-1/2} \overline{\mu}_{\lambda\kappa}^\xi$ converge in distribution as $\lambda \rightarrow \infty$ to those of a generalized mean-zero Gaussian field with covariance kernel

$$(f_1, f_2) \mapsto \int_{\mathbb{R}^d} f_1(x) f_2(x) V^\xi(\kappa(x)) \kappa(x) dx.$$

The above results capture the weak law of large numbers and the Gaussian limit behavior of the re-normalized measures $\lambda^{-1/2} \overline{\mu}_{\lambda\kappa}^\xi$.

1.4. Estimates on deviation probabilities

Starting from the known results it is natural to investigate the asymptotics of *deviation probabilities* on a scale larger than that of the central limit theorem. To this end we consider a fixed test-function $f \in \mathcal{B}(\mathbb{R}^d)$ and strive to get precise information on bounds of the relative error

$$\frac{\mathbb{P}(\langle f, \overline{\mu}_{\lambda\kappa}^\xi \rangle \geq x)}{1 - \Phi(x/\sigma_\lambda[f])}, \text{ as well as } \frac{\mathbb{P}(\langle f, \overline{\mu}_{\lambda\kappa}^\xi \rangle \leq -x)}{\Phi(-x/\sigma_\lambda[f])}, \quad x > 0, \quad (1.3)$$

where, as in the preceding Subsection 1.3, $\sigma_\lambda^2[f]$ denotes the variance and where, as usual,

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-t^2/2} dt$$

is the distribution function of the standard normal. In particular, we are interested in conditions under which the relative error (1.3) converges to 1 uniformly in the interval $0 \leq x \leq F(\lambda)$, where $F(\lambda)$ is a nondecreasing function such that $F(\lambda) \rightarrow \infty$. Of course $F(\lambda)$ will depend not only on λ but also on other characteristics of our models, in particular their dimensionality. For the sake of readability the dependence on other quantities is suppressed in all of our notation.

Since we will refine the cumulant expansion method of [3] to establish more precise rates of growth on the cumulants, in both their scale parameter and their order, we will be able to apply a powerful and general lemma on deviation probabilities due to Rudzkis, Saulis and Statulevičius [26], whose version specialized for our purposes is stated as Lemma 3.4 in the sequel for the convenience of the reader.

Using the afore-mentioned techniques we shall establish the following estimate for deviation probabilities of the centered empirical measures $\overline{\mu}_{\lambda\kappa}^\xi$:

Theorem 1.1. Suppose that ξ satisfies Assumptions $(*)$ and take $f \in \mathcal{B}(\mathbb{R}^d)$.

(1) Suppose that $Q_\kappa^\xi(f) > 0$. Then, for $\lambda \geq \lambda_0$ and $0 \leq x \leq C_1 \lambda^{(4+2d)/(6+4d)}$, we have:

$$\left| \log \frac{\mathbb{P}(\langle f, \bar{\mu}_{\lambda\kappa}^\xi \rangle \geq x)}{1 - \Phi(x/\sigma_\lambda[f])} \right| \leq C_2 \left(\frac{1}{\lambda^{1/(6+4d)}} + \frac{x^3}{\lambda^{(10+6d)/(6+4d)}} \right), \quad (1.4)$$

$$\left| \log \frac{\mathbb{P}(\langle f, \bar{\mu}_{\lambda\kappa}^\xi \rangle \leq -x)}{\Phi(-x/\sigma_\lambda[f])} \right| \leq C_2 \left(\frac{1}{\lambda^{1/(6+4d)}} + \frac{x^3}{\lambda^{(10+6d)/(6+4d)}} \right). \quad (1.5)$$

(2) Suppose that $0 < \sigma_\lambda[f] \leq C_3 \lambda^{1/2}$. Then, for all $x \geq 0$, we have:

$$\mathbb{P}(\pm \langle f, \bar{\mu}_{\lambda\kappa}^\xi \rangle \geq x) \leq \exp \left(- \min \left\{ C_4 \frac{x^2}{\sigma_\lambda^2[f]}, C_5 x^{1/(2+d)}, C_6 \left(\frac{x^3}{\lambda} \right)^{1/(3+d)} \right\} \right). \quad (1.6)$$

The constants $C_1 - C_6$ and λ_0 only depend on f , κ and ξ .

Remark 1.2. The second part of the theorem above is especially useful for degenerate cases, i. e., $Q_\kappa^\xi(f) = 0$.

1.5. Moderate deviation principles

It is natural to investigate the asymptotics of $(\bar{\mu}_{\lambda\kappa}^\xi)_\lambda$ on intermediate scales between those appearing in Gaussian and law of large numbers behavior. This leads us to moderate deviation principles (MDP). In this paper we are able to deduce moderate deviation principles from Theorem 1.1 for a *partial* intermediate regime for all strongly exponentially stabilizing and bounded ξ . Remark that in [1] moderate deviation principles were obtained for an essentially smaller set of examples, including the prototypical random sequential packing and some spatial birth-and-growth models as well as for empirical functionals of nearest neighbor graphs, but they were obtained on *every* intermediate scale.

We say that a family of probability measures $(\nu_\lambda)_\lambda$ on \mathcal{T} , which is a measurable as well as a topological space, obeys a large deviation principle (LDP) with speed α_λ and good rate function $I(\cdot): \mathcal{T} \rightarrow [0, \infty]$ as $\lambda \rightarrow \lambda_0$ if

- I is lower semi-continuous and has compact level sets $N_L := \{x \in \mathcal{T}: I(x) \leq L\}$, for every $L \in [0, \infty)$.
- For every measurable set Γ , we have:

$$- \inf_{x \in \overset{\circ}{\Gamma}} I(x) \leq \liminf_{\lambda \rightarrow \lambda_0} \frac{1}{\alpha_\lambda} \log \nu_\lambda(\Gamma) \leq \limsup_{\lambda \rightarrow \lambda_0} \frac{1}{\alpha_\lambda} \log \nu_\lambda(\Gamma) \leq - \inf_{x \in \bar{\Gamma}} I(x), \quad (1.7)$$

where $\overset{\circ}{\Gamma}$ denotes the topological interior of Γ and $\bar{\Gamma}$ denotes its closure.

Notice that we do not assume that the measures are Borel. In other words, open sets are not necessarily measurable.

Similarly we will say that a family of \mathcal{T} -valued random variables $(Y_\lambda)_\lambda$ obeys a large deviation principle with speed α_λ and good rate function $I(\cdot): \mathcal{T} \rightarrow [0, \infty]$ if the sequence of their distributions does. Formally a moderate deviation principle is nothing but an LDP. However, we will speak about a moderate deviation principle (MDP) for a sequence of random variables whenever the scaling of the corresponding random variables is between that of an ordinary Law of Large Numbers and that of a Central Limit Theorem.

We consider $\lambda \in (0, \infty)$, $\lambda \rightarrow \infty$ and let $(\alpha_\lambda)_{\lambda > 0}$ be such that

$$\lim_{\lambda \rightarrow \infty} \alpha_\lambda = \infty \quad \text{and} \quad \lim_{\lambda \rightarrow \infty} \alpha_\lambda \lambda^{-1/(6+4d)} = 0. \quad (1.8)$$

Under these assumptions first we obtain the following MDP for $\bar{\mu}_{\lambda\kappa}^\xi$:

Theorem 1.2. Suppose that ξ satisfies Assumptions $(*)$ and take $f \in \mathcal{B}(\mathbb{R}^d)$ with $Q_\kappa^\xi(f) > 0$. Then, for each $(\alpha_\lambda)_{\lambda>0}$ satisfying (1.8), the family of random variables $(\alpha_\lambda^{-1} \lambda^{-1/2} \langle f, \bar{\mu}_{\lambda\kappa}^\xi \rangle)_\lambda$ satisfies on \mathbb{R} the moderate deviation principle with speed α_λ^2 and good rate function

$$I_{\kappa;f}^\xi(t) := \frac{t^2}{2Q_\kappa^\xi(f)} \quad (1.9)$$

(recalling our convention on division by zero from the end of Subsection 1.2).

The next result is a MDP on the level of *measures*. Let us denote by $\mathcal{M}(\mathbb{R}^d)$ the real vector space of finite signed measures on \mathbb{R}^d . Equip $\mathcal{M}(\mathbb{R}^d)$ with the τ -topology generated by the sets:

$$U_{f,x,\delta} := \{ \nu \in \mathcal{M}(\mathbb{R}^d) ; |\langle f, \nu \rangle - x| < \delta \},$$

where $f \in \mathcal{B}(\mathbb{R}^d)$, $x \in \mathbb{R}$ and $\delta > 0$. It is well known that since the collection of linear functionals $\{ \nu \mapsto \langle f, \nu \rangle ; f \in \mathcal{B}(\mathbb{R}^d) \}$ is separating in $\mathcal{M}(\mathbb{R}^d)$, this topology makes $\mathcal{M}(\mathbb{R}^d)$ into a locally convex, Hausdorff topological vector space, whose topological dual is the preceding collection, hereafter identified with $\mathcal{B}(\mathbb{R}^d)$. With this notation we establish the following *measure-level* MDP for $\bar{\mu}_{\lambda\kappa}^\xi$:

Theorem 1.3. Suppose that ξ satisfies Assumptions $(*)$. Then for any family $(\alpha_\lambda)_{\lambda>0}$ satisfying (1.8), the family

$$(\alpha_\lambda^{-1} \lambda^{-1/2} \bar{\mu}_{\lambda\kappa}^\xi)_\lambda$$

satisfies the MDP on $\mathcal{M}(\mathbb{R}^d)$, endowed with the τ -topology, with speed α_λ^2 and the convex, good rate function given by

$$I_\kappa^\xi(\nu) := \frac{1}{2} Q_\kappa^\xi \left(\frac{d\nu}{V^\xi(\kappa(x)) \kappa(x) dx} \right) \quad (1.10)$$

if $\nu \in \mathcal{M}(\mathbb{R}^d)$ is absolutely continuous with respect to $V^\xi(\kappa(x)) \kappa(x) dx$, and by $I_\kappa^\xi(\nu) := +\infty$ otherwise.

Remark 1.3. The τ -topology, for which the MDP in Theorem 1.3 is stated, is stronger than the weak topology, which is generated only by bounded continuous test functions and which the corresponding Theorem 2.2 in [1] is based on.

2. Applications

We provide here applications of our deviation bounds and moderate deviation principles to functionals in geometric probability, including functionals of random sequential packing models and functionals of graphs in computational geometry. The following examples have been considered in detail in the context of central limit theorems [21, 3, 24] and laws of large numbers [22, 23]. Some of these have also been considered in [1] where the corresponding moderate deviation principles have been established. The detailed discussion of where our present paper improves and generalizes over [1] is provided on a case-by-case basis below.

2.1. Packing

2.1.1. Random sequential packing

The following prototypical random sequential packing/adsorption (RSA) model arises in diverse disciplines, including physical, chemical, and biological processes. See [22] for a discussion of the many applications, the many references, and also a discussion of previous mathematical analysis. In one dimension, this model is often referred to as the Rényi car parking model [25].

Consider a point configuration \mathcal{X} and to each $x \in \mathcal{X}$ attach a unit ball centered at x . Moreover, to all points in \mathcal{X} attach i.i.d. uniform time marks taking values in some finite time interval $[0, T]$, $T > 0$. This establishes a *chronological* order on the points of \mathcal{X} . Declare the first point in this ordering *accepted* and proceed recursively, each time accepting the consecutive point if the ball it carries does not overlap the previously accepted (packed) balls and rejecting it otherwise. The functional $\xi(x, \mathcal{X})$ is defined to be 1 if

the ball centered at x has been accepted and 0 otherwise. This defines the prototypical random sequential packing/adsorption (RSA) process.

It is known that the so-defined packing functional ξ is strongly exponentially stabilizing, see [22]. The boundedness of ξ follows by its definition. Consequently, ξ satisfies the conditions and hence also the conclusions of Theorems 1.1, 1.2 and 1.3.

Compared to the results in [1], the novelty here is that we provide more explicit bounds in Theorem 1.1. The counterparts of Theorems 1.2 and 1.3 are already established in [1].

Our present results add to existing central limit theorems [12, 2, 3, 22], weak laws of large numbers [8, 22, 23] and large deviations [28] for random packing functionals.

2.1.2. Spatial Birth-Growth Models

Consider the following generalization of the above basic RSA model: the balls attached to subsequent independently time-marked points, further interpreted as *particles*, are allowed to have their initial radii bounded random i.i.d. rather than fixed. Moreover, at the moment of its birth each particle begins to *grow* radially with constant speed v until it hits another particle or reaches a certain maximal admissible size – in both these cases it stops growing. In analogy to the basic RSA, a particle is accepted if it does not overlap any previously accepted one and is discarded otherwise. The functional of interest is again given by $\xi(x, \mathcal{X}) = 1$ if the particle centered at x has been accepted and 0 otherwise. This model, going also under the name of the Johnson-Mehl growth process in the particular case where the initial radii are 0, has attracted a lot of interest in the literature, see [22, 3] and the references therein.

It is known, see *ibidem*, that the so-defined functional ξ is strongly exponentially stabilizing. The boundedness of ξ follows by its definition. Consequently, ξ satisfies the conditions and hence also the conclusions of Theorems 1.1, 1.2 and 1.3.

In [1] we were only able to treat the spatial birth and growth models under an unnatural positive lower bound for initial particle sizes, which excluded for instance the crucial Johnson-Mehl set-up. Here this condition is no more required.

Our present results add to the existing central limit theorems [7, 22, 3] as well as to the large deviation principle [28].

2.2. Random graphs

2.2.1. k -nearest Neighbors Graphs

Fix a positive integer k . By the k -nearest neighbor graph $\text{NG}^{\rightarrow}(\mathcal{X}; k)$ on a locally finite point configuration \mathcal{X} we mean the directed graph where an edge is present from x to y in \mathcal{X} whenever y is among the k nearest neighbors of x . We also consider its undirected version $\text{NG}^{\leftrightarrow}(\mathcal{X})$ arising by forgetting the direction of edges and collapsing possible double edges into single ones. A broad family of functionals, further referred to as *bounded edge length functionals*, arise as

$$\xi(x, \mathcal{X}) := \sum_{e \in \text{Edges}(x; \text{NG}^*(\mathcal{X}))} \phi(|e|),$$

with $\text{Edges}(x; \text{NG}^*(\mathcal{X}))$ standing for the collection of edges outgoing from x in the graph $\text{NG}^{\rightarrow}(x; \mathcal{X})$ or $\text{NG}^{\leftrightarrow}(x; \mathcal{X})$ and where ϕ is a non-negative and bounded real function.

It is known, see e.g. [22, 3], that the so-defined functional ξ is strongly exponentially stabilizing. The boundedness of ξ follows by its definition and, for $\text{NG}^{\leftrightarrow}$, finite dimensionality. Consequently, ξ satisfies the conditions and hence also the conclusions of Theorems 1.1, 1.2 and 1.3.

Compared to [1] here our theorems are much more general. In fact, in [1] we were only able to deal with the *empirical functionals of nearest neighbor graphs* which either admit the above representation with 0–1-valued ϕ indicating whether the edge length exceeds a certain threshold, or are defined as an indicator function of some event involving the degree of the graph at x and possibly also the edge length, such as ‘the total length of edges incident to x exceeds a certain multiplicity of the graph degree of x ’ etc.

Our present results add to existing laws of large numbers and central limit theorems, see Chapter 4 of [17] as well as [3]. Recently many explicit calculations for the first order characteristics of nearest neighbor graphs have been provided by [30].

2.2.2. Voronoi cells and Delaunay Graphs

Given a locally finite set $\mathcal{X} \subset \mathbb{R}^d$ and $x \in \mathcal{X}$, the locus of points closer to x than to any other point in \mathcal{X} is called the *Voronoi cell* centered at x . Denote it by $\mathcal{C}(x, \mathcal{X})$. The graph on vertex set \mathcal{X} in which each pair of adjacent cell centers is connected by an edge is called the *Delaunay graph* on \mathcal{X} . Let $\xi(x, \mathcal{X})$ be a bounded functional of the form $\xi(x, \mathcal{X}) := \psi(\mathcal{C}(x, \mathcal{X}))$ where ψ is a bounded real-valued function defined on convex bodies in \mathbb{R}^d , natural examples including bounded functionals of the perimeter, area and surface measures of $\mathcal{C}(x, \mathcal{X})$. Alternatively, we can also let $\xi(x, \mathcal{X})$ be defined by a bounded real-valued functional depending on x and the collection of its outgoing Delaunay edges.

It is known, see e.g. [3], that the so-defined functionals ξ are strongly exponentially stabilizing. Boundedness of ξ is clear. Such ξ satisfy the conditions and hence also the conclusions of Theorems 1.1, 1.2 and 1.3.

The Voronoi cells and Delaunay graphs were not considered in [1].

Our present results add to the laws of large numbers and central limit theorems of [23] and [21] respectively.

2.2.3. Sphere of Influence Graph

Given a locally finite set $\mathcal{X} \subset \mathbb{R}^d$, the sphere of influence graph $\text{SIG}(\mathcal{X})$ is a graph with vertex set \mathcal{X} , constructed as follows: for each $x \in \mathcal{X}$ let $B(x)$ be a ball around x with radius equal to $\min_{y \in \mathcal{X} \setminus \{x\}} \{|y - x|\}$. Then $B(x)$ is called the sphere of influence of x . Draw an edge between x and y iff the balls $B(x)$ and $B(y)$ overlap. The collection of such edges is the sphere of influence graph (SIG) on \mathcal{X} and is denoted by $\text{SIG}(\mathcal{X})$.

The functionals ξ of our interest are:

1. The reciprocal of the connected component cardinality at x in $\text{SIG}(\mathcal{X})$. Note that in this case, $\sum_{x \in \mathcal{X}} \xi(x, \mathcal{X})$ is simply the number of connected components of $\text{SIG}(\mathcal{X})$.
2. Bounded real-valued functionals depending on x and the collection of its outgoing SIG edges.

Again, it is known, see [21, 3], that the so-defined functionals ξ are strongly exponentially stabilizing. Boundedness of such ξ is clear. Thus, such ξ satisfy the conditions and hence also the conclusions of Theorems 1.1, 1.2 and 1.3.

The sphere of influence graphs have not been considered in [1].

Our present results add to central limit theorems of [21, 3].

3. Proofs of the results

3.1. The method of cumulants

We will refine the method of cumulants and cluster measures as developed in [3] in the context of the central limit theorem. We recall the formal definition of cumulants in the context specified for our purposes. Taking $f \in \mathcal{B}(\mathbb{R}^d)$, expand $\mathbb{E} \exp \left(\langle f, \mu_{\lambda\kappa}^\xi \rangle \right)$ in a power series in f as follows:

$$\mathbb{E} \exp \left(\langle f, \mu_{\lambda\kappa}^\xi \rangle \right) = 1 + \sum_{k=1}^{\infty} \frac{\langle f^k, M_\lambda^k \rangle}{k!}, \quad (3.1)$$

where $f^k: (\mathbb{R}^d)^k \rightarrow \mathbb{R}$, $k = 1, 2, \dots$ is given by $f^k(v_1, \dots, v_k) = f(v_1) \cdot \dots \cdot f(v_k)$. Note that $M_\lambda^k := M_\lambda^k(\kappa)$ is a measure on $(\mathbb{R}^d)^k$, the k -th moment measure of $\mu_{\lambda\kappa}^\xi$ (p. 130 of [9]). Both the existence of the moment measures and the convergence of the series (3.1) are direct consequences of the boundedness of ξ .

The moment measures can be expressed in terms of singular measures as follows. First, define the singular differential $\bar{d}[g]v$ as:

$$\int_{(\mathbb{R}^d)^k} F(v_1, v_2, \dots, v_k) \bar{d}[g](v_1, \dots, v_k) := \int_{\mathbb{R}^d} F(x, x, \dots, x) g(x) dx$$

Next, for $v = (v_1, \dots, v_k)$ running over $(\mathbb{R}^d)^k$, put:

$$\tilde{d}[g]v := \sum_{L_1, \dots, L_p} \bar{d}[g]v_{L_1} \cdots \bar{d}[g]v_{L_p},$$

where $v_L := (v_l)_{l \in L}$ and where the sum ranges over all unordered partitions of the set $\{1, \dots, k\}$. By decomposing the moment measure M_λ^k according to formula (4.4) on p. 83 of [15], followed by conditioning on the marks and repeated disintegration (Palm theory for the Poisson point process), we obtain:

$$M_\lambda^k(dv) = m_\lambda(v) \tilde{d}[\lambda\kappa]v, \quad (3.2)$$

where the Radon-Nikodym derivative m_λ is given by

$$m_\lambda(v_1, \dots, v_k) := \mathbb{E} \left[\prod_{i=1}^k \xi_\lambda(v_i; \mathcal{P}_{\lambda\kappa} \cup \{v_j\}_{j=1}^k) \right]. \quad (3.3)$$

For each fixed k , the mixed moment on the right hand side of (3.3) is bounded in absolute value by C_ξ^k uniformly in λ since $|\xi| \leq C_\xi$.

Expanding the logarithm of the Laplace transform in a formal power series gives

$$\log \left[1 + \sum_{k=1}^{\infty} \frac{\langle f^k, M_\lambda^k \rangle}{k!} \right] = \sum_{l=1}^{\infty} \frac{\langle f^l, c_\lambda^l \rangle}{l!}; \quad (3.4)$$

the signed measures c_λ^l are *cumulant measures* [16] of $\mu_{\lambda\kappa}^\xi$. In the sequel we shall also be interested in the cumulant measures \bar{c}_λ^l of the centered measures $\bar{\mu}_{\lambda\kappa}^\xi$, which are easily checked to satisfy $\bar{c}_\lambda^1 = 0$ and $\bar{c}_\lambda^l = c_\lambda^l$ for $l \geq 2$. To establish Theorem 1.1, we shall develop growth bounds on $\langle f^l, c_\lambda^l \rangle$ in terms of both the scale parameter λ and the order l .

Note that we do not require the convergence of the formal series in (3.4). On the other hand, the existence of all cumulants c_λ^l , $l = 1, 2, \dots$ follows from the existence of all moments, in view of the representation

$$c_\lambda^l = \sum_{L_1, \dots, L_p} (-1)^{p-1} (p-1)! M_\lambda^{L_1} \cdots M_\lambda^{L_p}$$

with $M_\lambda^{L_i}$ standing for a copy of the moment measure $M_\lambda^{|L_i|}$ on the product space $(\mathbb{R}^d)^{L_i}$ (see below for formal details) and where L_1, \dots, L_p again ranges over all unordered partitions of the set $\{1, \dots, l\}$ (see p. 30 of [16]). The first cumulant measure coincides with the expectation measure and the second cumulant measure coincides with the covariance measure.

We will sometimes shorten notation and write M^k, m and c^l, \bar{c}^l instead of M_λ^k, m_λ and $c_\lambda^l, \bar{c}_\lambda^l$.

It is important to emphasize the following relationship between the cumulant measures of random measures and (mixed) cumulants of random variables. Consider general random variables Y_1, \dots, Y_k and construct a random measure $\nu := \sum_{i=1}^k Y_i \delta_i$. Defining the k -th cumulant measure $c^k[\nu]$ for ν in full analogy with the construction of c_λ^k for $\mu_{\lambda\kappa}^\xi$ we observe that the mixed k -th cumulant $c[Y_1, \dots, Y_k]$ of (Y_1, \dots, Y_k) coincides with $c^k[\nu](\{(1, 2, \dots, k)\})$. Moreover, the usual k -th cumulant $c^k[Y]$ of Y is clearly equal to $c \underbrace{[Y, \dots, Y]}_k$.

3.2. Cluster measures and bounds on the cumulants

For any subset L of the positive integers, we denote by $(\mathbb{R}^d)^L$ the product of $|L|$ copies of \mathbb{R}^d , i. e.

$$(\mathbb{R}^d)^L := \prod_{i \in L} \mathbb{R}^d.$$

A *cluster measure* $U^{S,T}$ on $(\mathbb{R}^d)^S \times (\mathbb{R}^d)^T$ for non-empty disjoint $S, T \subseteq \{1, 2, \dots\}$, is defined by

$$U^{S,T}(A \times B) = M^{S \cup T}(A \times B) - M^S(A) M^T(B)$$

for all Borel sets A and B in $(\mathbb{R}^d)^S$ and $(\mathbb{R}^d)^T$, respectively.

For L a subset of $\{1, 2, \dots\}$, by c^L we mean the cumulant

$$c^L = \sum_{L_1, \dots, L_p} (-1)^{p-1} (p-1)! M^{L_1} \dots M^{L_p}, \quad (3.5)$$

where L_1, \dots, L_p ranges over all unordered partitions of L . In fact, c^L is just a copy of c^l , where $l = |L|$. We use the notation c^l and $c^{\{1, \dots, l\}}$ interchangeably.

Lemma 3.1. *Let (S, T) be a non-trivial partition of $\{1, 2, \dots, k\}$. The cumulant c^k admits the representation:*

$$c^k = \sum_{S', T'; K_1, \dots, K_s} a_{S', T'; K_1, \dots, K_s} U^{S', T'} M^{K_1} M^{K_2} \dots M^{K_s},$$

where, in each summand, S', T', K_1, \dots, K_s is a partition of $\{1, \dots, k\}$ with $S' \subseteq S$ and $T' \subseteq T$, and where:

$$\sum_{S', T'; K_1, \dots, K_s} |a_{S', T'; K_1, \dots, K_s}| \leq k^k. \quad (3.6)$$

Proof. Starting from (3.5), we first note that each moment measure M^{L_i} with $S' := L_i \cap S \neq \emptyset$ and $T' := L_i \cap T \neq \emptyset$, can be expressed as $U^{S', T'} + M^{S'} M^{T'}$. Thus, each product $M^{L_1} \dots M^{L_p}$ can be expressed as a sum of at most p terms of the form $U^{S', T'} M^{K_1} \dots M^{K_s}$, plus a product $M^{Z_1} M^{Z_2} \dots M^{Z_t}$, where Z_1, \dots, Z_t is a partition of the set $\{1, \dots, k\}$ consisting of all non-empty intersections of the sets L_1, \dots, L_p with S and T . Summing up, we obtain:

$$c^k = \sum_{S', T'; K_1, \dots, K_s} a_{S', T'; K_1, \dots, K_s} U^{S', T'} M^{K_1} M^{K_2} \dots M^{K_s} + \sum_{Z_1, \dots, Z_t} b_{Z_1, \dots, Z_t} M^{Z_1} \dots M^{Z_t},$$

where the second sum ranges over all unordered partitions of $\{1, \dots, k\}$. Clearly,

$$\sum_{S', T'; K_1, \dots, K_s} |a_{S', T'; K_1, \dots, K_s}| \leq \sum_{L_1, \dots, L_p} p!, \quad (3.7)$$

where the sum on the right-hand side again ranges over all unordered partitions. This sum is exactly the number of all ordered partitions, which does not exceed k^k . Finally, the coefficients b_{Z_1, \dots, Z_t} satisfy:

$$b_{Z_1, \dots, Z_t} = \sum_{p=1}^k (-1)^p (p-1)! R_p(Z_1, \dots, Z_t), \quad (3.8)$$

where $R_p(Z_1, \dots, Z_t)$ is the number of all unordered partitions L_1, \dots, L_p , such that Z_1, \dots, Z_t are exactly all non-empty intersections of the sets L_1, \dots, L_p with S and T . But from part (2) of Lemma 2 in Gorchakov [14], it is can be deduced that the sum in (3.8) vanishes. This completes the proof. \square

To obtain our crucial bound on cumulant integrals stated in Lemma 3.3 below, we shall need a slightly refined version of Lemma 5.2 in [3], namely

Lemma 3.2. *Suppose that ξ satisfies Assumptions $(*)$ and let S and T be non-empty disjoint index sets. Then we have for some $A, B, \beta > 0$ depending only on the decay rate of the stabilization for ξ and $\|\kappa\|_\infty$,*

$$|m_\lambda(v_{S \cup T}) - m_\lambda(v_S) m_\lambda(v_T)| \leq A k B^k C_\xi^k \exp(-\beta \delta \lambda^{1/d}),$$

where $k = |S \cup T|$, $v_J := (v_j)_{j \in J}$ and where $\delta := d(v_S, v_T) := \min_{s \in S, t \in T} \|v_s - v_t\|$ is the separation between the sets arising from v_S and v_T .

Proof. We follow the proof of Lemma 5.2 in [3]. Putting:

$$X := \prod_{s \in S} \xi_\lambda(v_s; \mathcal{P}_{\lambda\kappa} \cup \{v_j\}_{j \in S}), \quad Y := \prod_{t \in T} \xi_\lambda(v_t; \mathcal{P}_{\lambda\kappa} \cup \{v_j\}_{j \in T}), \quad W := \prod_{r \in S \cup T} \xi_\lambda(v_r; \mathcal{P}_{\lambda\kappa} \cup \{v_j\}_{j \in S \cup T})$$

and:

$$X_\delta := \prod_{s \in S} \xi_\lambda(v_s; (\mathcal{P}_{\lambda\kappa} \cup \{v_j\}_{j \in S}) \cap B_{\delta/2}(v_s)), \quad Y_\delta := \prod_{t \in T} \xi_\lambda(v_t; (\mathcal{P}_{\lambda\kappa} \cup \{v_j\}_{j \in T}) \cap B_{\delta/2}(v_t)), \quad (3.9)$$

$$W_\delta := \prod_{r \in S \cup T} \xi_\lambda(v_r; (\mathcal{P}_{\lambda\kappa} \cup \{v_j\}_{j \in S \cup T}) \cap B_{\delta/2}(v_r)), \quad (3.10)$$

we may write:

$$\begin{aligned} m_\lambda(v_{S \cup T}) - m_\lambda(v_S) m_\lambda(v_T) &= \mathbb{E} W - \mathbb{E} X \mathbb{E} Y = \\ &= \mathbb{E} W_\delta - \mathbb{E} X_\delta \mathbb{E} Y_\delta + \mathbb{E}(W - W_\delta) - \mathbb{E} X_\delta \mathbb{E}(Y - Y_\delta) - \mathbb{E}(X - X_\delta) \mathbb{E} Y. \end{aligned}$$

Since the balls $B_{\delta/2}(v_s)$, $s \in S$, do not intersect with the balls $B_{\delta/2}(v_t)$, $t \in T$, and because of independence, we have $\mathbb{E} W_\delta = \mathbb{E} X_\delta Y_\delta = \mathbb{E} X_\delta \mathbb{E} Y_\delta$. The other terms can be estimated by the obvious bounds $|X| \leq C_\xi^{|S|}$, $|Y| \leq C_\xi^{|T|}$, and by $|W - W_\delta| \leq 2C_\xi^k \mathbf{1}(N_{S \cup T})$, $|X - X_\delta| \leq 2C_\xi^{|S|} \mathbf{1}(N_S)$ and $|Y - Y_\delta| \leq 2C_\xi^{|T|} \mathbf{1}(N_T)$, where N_J denotes the event that the radius of stabilization at at least one v_j , $j \in J$, is greater or equal to $\delta/2$. Recalling Definitions 1.4 and 1.5, we find that $\mathbb{P}(N_J) \leq |J|LB^k \exp(-\alpha\lambda^{1/d}\delta/2)$ and consequently $|m_\lambda(v_{S \cup T}) - m_\lambda(v_S) m_\lambda(v_T)| \leq 4Lk B^k C_\xi^k \exp(-\alpha\delta\lambda^{1/d}/2)$, which proves the desired result. \square

This puts us in a position to formulate our cumulant bound lemma.

Lemma 3.3. *Suppose that ξ satisfies Assumptions (*). Then there are constants $A, B < \infty$ depending only on ξ and κ , such that for all bounded measurable functions $F: (\mathbb{R}^d)^k \rightarrow \mathbb{R}$, all $k = 2, 3, \dots$ and all $\lambda > 0$, we have:*

$$|\langle F, c_\lambda^k \rangle| \leq AB^k \|F\|_\infty (k!)^{d+2} \lambda.$$

Proof. First, we introduce some notation. Recalling the definition of v_J and $d(v_S, v_T)$ from Lemma 3.2, we define for each $v \in (\mathbb{R}^d)^k$:

$$\delta(v) := \max\{d(v_S, v_T) : (S, T) \text{ is a partition of } \{1, \dots, k\}\}. \quad (3.11)$$

Denoting by $\Delta = \{(x, x, \dots, x) : x \in \mathbb{R}^d\}$ the diagonal in $(\mathbb{R}^d)^k$, observe that $(\mathbb{R}^d)^k \setminus \Delta$ can be divided into a disjoint union of sets $\sigma(\{S, T\})$, where $\{S, T\}$ is an unordered partition of the set $\{1, \dots, k\}$ and where $\delta(v) = d(v_S, v_T)$ for all $v \in \sigma(\{S, T\})$. In other words, for each $v \in (\mathbb{R}^d)^k \setminus \Delta$, we choose the partition where the maximum in (3.11) is attained. Thus we may write:

$$\langle F, c_\lambda^k \rangle = \int_{(\mathbb{R}^d)^k} F \, dc_\lambda^k = \int_\Delta F \, dc_\lambda^k + \sum_{S, T} \int_{\sigma(\{S, T\})} F \, dc_\lambda^k, \quad (3.12)$$

where the sum ranges over all unordered partitions of $\{1, \dots, k\}$ into two sets. For the first term, we have:

$$\int_\Delta F \, dc_\lambda^k = \sum_{L_1, \dots, L_p} (-1)^{p-1} (p-1)! \int_\Delta F \, d(M_\lambda^{L_1} \dots M_\lambda^{L_p}),$$

where, as usual, the sum ranges over all unordered partitions of $\{1, \dots, k\}$. However, from (3.2) and (3.3), we find that only the partition into one single set can give a non-zero term. Therefore,

$$\int_\Delta F \, dc_\lambda^k = \int_\Delta F(v) m_\lambda(v) \, d[\lambda\kappa]v = \lambda \int_{\mathbb{R}^d} F(x, \dots, x) \mathbb{E}[\xi_\lambda(x; \mathcal{P}_{\lambda\kappa})]^k \kappa(x) \, dx.$$

Using the boundedness of ξ and the fact that κ is a probability density, we find that:

$$\left| \int_{\Delta} F \, dc_{\lambda}^k \right| \leq C_{\xi}^k \|F\|_{\infty} \lambda. \quad (3.13)$$

The terms in the remaining sum in (3.12) can be decomposed in view of Lemma 3.1:

$$\int_{\sigma(\{S,T\})} F \, dc_{\lambda}^k = \sum_{S', T'; K_1, \dots, K_s} a_{S', T'; K_1, \dots, K_s} \int_{\sigma(\{S,T\})} F \, d(U_{\lambda}^{S', T'} M_{\lambda}^{K_1} \dots M_{\lambda}^{K_s}).$$

Plugging into (3.2), we compute:

$$\begin{aligned} \int_{\sigma(\{S,T\})} F \, d(U_{\lambda}^{S', T'} M_{\lambda}^{K_1} \dots M_{\lambda}^{K_s}) &= \int_{\sigma(\{S,T\})} [m_{\lambda}(v_{S' \cup T'}) - m_{\lambda}(m_{S'}) m_{\lambda}(v_{T'})] m_{\lambda}(v_{K_1}) \dots m_{\lambda}(v_{K_s}) \times \\ &\quad \times \tilde{d}[\lambda \kappa]_{v_{S'}} \tilde{d}[\lambda \kappa]_{v_{T'}} \tilde{d}[\lambda \kappa]_{v_{K_1}} \dots \tilde{d}[\lambda \kappa]_{v_{K_s}}. \end{aligned}$$

Making use of Lemma 3.2, we find that for some A_1 , B_1 and $\beta > 0$:

$$\begin{aligned} \left| \int_{\sigma(\{S,T\})} F \, d(U_{\lambda}^{S', T'} M_{\lambda}^{K_1} \dots M_{\lambda}^{K_s}) \right| &\leq A_1 k B_1^k C_{\xi}^k \|F\|_{\infty} \int_{\sigma(\{S,T\})} \exp(-\beta d(v_{S'}, v_{T'}) \lambda^{1/d}) \times \\ &\quad \times \tilde{d}[\lambda \kappa]_{v_{S'}} \tilde{d}[\lambda \kappa]_{v_{T'}} \tilde{d}[\lambda \kappa]_{v_{K_1}} \dots \tilde{d}[\lambda \kappa]_{v_{K_s}} \leq \\ &\leq A_1 k B_1^k C_{\xi}^k \|F\|_{\infty} \int_{\sigma(\{S,T\})} \exp(-\beta d(v_S, v_T) \lambda^{1/d}) \tilde{d}[\lambda \kappa]_v = \\ &= A_1 k B_1^k C_{\xi}^k \|F\|_{\infty} \int_{\sigma(\{S,T\})} \exp(-\beta \delta(v) \lambda^{1/d}) \tilde{d}[\lambda \kappa]_v. \end{aligned}$$

Summing up and applying (3.6), we obtain:

$$\left| \sum_{S,T} \int_{\sigma(\{S,T\})} F \, dc_{\lambda}^k \right| \leq A_1 B_1^k C_{\xi}^k k^{k+1} \|F\|_{\infty} \int_{(\mathbb{R}^d)^k} \exp(-\beta \delta(v) \lambda^{1/d}) \tilde{d}[\lambda \kappa]_v. \quad (3.14)$$

Recalling the definition of the differential \tilde{d} , we expand:

$$\begin{aligned} \int_{(\mathbb{R}^d)^k} \exp(-\beta \delta(v) \lambda^{1/d}) \tilde{d}[\lambda \kappa]_v &= \sum_{L_1, \dots, L_p} \int_{(\mathbb{R}^d)^k} \exp(-\beta \delta(v) \lambda^{1/d}) \tilde{d}[\lambda \kappa]_{v_{L_1}} \dots \tilde{d}[\lambda \kappa]_{v_{L_p}} = \\ &= \sum_{L_1, \dots, L_p} \lambda^p \int_{(\mathbb{R}^d)^p} \exp(-\beta \delta(x) \lambda^{1/d}) \kappa^p(x) \, dx. \end{aligned} \quad (3.15)$$

Now write:

$$\begin{aligned} \int_{(\mathbb{R}^d)^p} \exp(-\beta \delta(x) \lambda^{1/d}) \kappa^p(x) \, dx &= \beta \lambda^{1/d} \int_{(\mathbb{R}^d)^p} \int_{\delta(x)}^{\infty} e^{-\beta \lambda^{1/d} t} \, dt \, \kappa^p(x) \, dx = \\ &= \beta \lambda^{1/d} \int_0^{\infty} \int_{\delta(x) < t} \kappa^p(x) \, dx \, e^{-\beta \lambda^{1/d} t} \, dt. \end{aligned}$$

If $\delta(x) < t$, there is no separation with respect to the distance t , so that there exists a tree D on vertices $\{1, \dots, p\}$, such that for any two adjacent indices i and j , v_i and v_j are closer than t . Denote this by $x \propto_t D$.

Thus,

$$\begin{aligned}
\int_{\delta(x) < t} \kappa^p(x) dx &\leq \sum_D \int_{x \propto_t D} \kappa^p(x) dx = \\
&= \sum_D \int_{\mathbb{R}^d} \int_{(y,z) \propto_t D} \kappa^{p-1}(z) dz \kappa(x) dx \leq \\
&\leq \|\kappa\|_\infty^{p-1} \sum_D \int_{\mathbb{R}^d} \text{vol}_{(p-1)d}(\{z \in (\mathbb{R}^d)^{p-1}; (y,z) \propto_t D\}) \kappa(y) dy,
\end{aligned}$$

where the sum ranges over all trees on vertices $\{1, \dots, p\}$. Noting that:

$$\text{vol}_{(p-1)d}(\{z \in (\mathbb{R}^d)^{p-1}; (y,z) \propto_t D\}) \leq t^{(p-1)d} V_d^{p-1},$$

where V_d here denotes the volume of the d -dimensional unit ball, and recalling that κ is a probability density and that there are exactly p^{p-2} trees on the vertex set $\{1, \dots, p\}$, we have:

$$\int_{\delta(x) < t} \kappa^p(x) dx \leq p^{p-2} t^{(p-1)d} V_d^{p-1} \|\kappa\|_\infty^{p-1}.$$

Integration yields:

$$\begin{aligned}
\int_{(\mathbb{R}^d)^p} \exp(-\beta \delta(x) \lambda^{1/d}) \kappa^p(x) dx &\leq \beta \lambda^{1/d} p^{p-2} V_d^{p-1} \|\kappa\|_\infty^{p-1} \int_0^\infty t^{(p-1)d} e^{-\beta \lambda^{1/d} t} dt = \\
&= \frac{p^{p-2} V_d^{p-1} \|\kappa\|_\infty^{p-1}}{\beta^{(p-1)d} \lambda^{p-1}} [(p-1)d]!.
\end{aligned}$$

Plugging this into (3.15), we find that:

$$\begin{aligned}
\int_{(\mathbb{R}^d)^k} \exp(-\beta \delta(v) \lambda^{1/d}) \tilde{d}[\lambda \kappa] v &\leq \lambda \left(\frac{V_d \|\kappa\|_\infty}{\beta^d} \right)^{p-1} \sum_{L_1, \dots, L_p} p^{p-2} [(p-1)d]! \leq \\
&\leq \lambda \left(\frac{V_d \|\kappa\|_\infty}{\beta^d} \right)^{p-1} [(k-1)d]! \sum_{L_1, \dots, L_p} p^{p-1}.
\end{aligned} \tag{3.16}$$

Now interpret p^{p-1} as the number of all mappings from the set of all unordered partitions of $\{1, \dots, k\}$ into itself which preserve the set containing 1. Each partition of $\{1, \dots, k\}$ and such a mapping φ can be assigned a mapping f on the set $\{1, \dots, k\}$, so that an element $i \in L$ is mapped to the smallest element in $\varphi(L)$. Notice that $f(1) = 1$ and it is not difficult to check that different partitions and different φ 's are assigned different f 's. Hence $\sum_{L_1, \dots, L_p} p^{p-1} \leq k^{k-1}$. Combining this fact with (3.12), (3.13), (3.14) and (3.16), we finally get:

$$|\langle F, c_\lambda^k \rangle| \leq A_2 B_2^k \|F\|_\infty k^{2k} [(k-1)d]! \lambda.$$

Applying the Stirling formula, the result follows. \square

3.3. Proof of bounds on deviation probabilities (Theorem 1.1)

As mentioned in Subsection 1.4, the proof of the result will be based on the estimation of the cumulants, applying the celebrated lemma of Rudzkis, Saulis and Statulevičius [26]. Consider a general random variable Y with finite absolute moments of all orders and recall that $c^k[Y]$ stands for the k -th cumulant of Y . Below we state a simplified form of the version of that lemma which appears as Lemma 2.3 on p. 18 of Saulis and Statulevičius [27]:

Lemma 3.4. *Let Y be as above. Suppose that $\mathbb{E}Y = 0$, $\text{Var}(Y) = 1$ and that there exists a $\gamma \geq 0$ and a $\Delta > 0$, such that the cumulants satisfy:*

$$|c^k[Y]| \leq \frac{(k!)^{1+\gamma}}{\Delta^{k-2}}, \quad k = 3, 4, \dots \quad (3.17)$$

Then the large deviation relations

$$\frac{\mathbb{P}(Y \geq y)}{1 - \Phi(y)} = \exp(L_\gamma(y)) \left(1 + \theta_1 \psi(y) \frac{y+1}{\Delta_\gamma} \right), \quad (3.18)$$

$$\frac{\mathbb{P}(Y \leq -y)}{\Phi(-y)} = \exp(L_\gamma(-y)) \left(1 + \theta_2 \psi(y) \frac{y+1}{\Delta_\gamma} \right) \quad (3.19)$$

hold in the interval $0 \leq y < \Delta_\gamma$. Here:

$$\Delta_\gamma = \frac{1}{6} \left(\frac{\sqrt{2}}{6} \Delta \right)^{1/(1+2\gamma)}, \quad (3.20)$$

$$\psi(y) = \frac{60[1 + 10\Delta_\gamma^2 \exp(-(1 - y/\Delta_\gamma)\sqrt{\Delta_\gamma})]}{1 - y/\Delta_\gamma}, \quad (3.21)$$

the quantities θ_1 and θ_2 belong to $[-1, 1]$ and the function $L_\gamma(y)$, which is closely related to the Cramér–Petrov series, satisfies:

$$|L_\gamma(y)| \leq \frac{|y|^3}{3\Delta_\gamma} \quad (3.22)$$

for all y with $|y| \leq \Delta_\gamma$.

The following weaker form of the preceding result will be used to prove the first part of Theorem 1.1:

Corollary 3.1. *Under the conditions of Lemma 3.4, there exist constants C_1 and C_2 depending only on γ , such that in the interval $0 \leq y \leq C_1 \Delta^{1/(1+2\gamma)}$, we can estimate:*

$$\left| \log \frac{\mathbb{P}(Y \geq y)}{1 - \Phi(y)} \right| \leq C_2 \frac{1 + y^3}{\Delta^{1/(1+2\gamma)}}, \quad (3.23)$$

$$\left| \log \frac{\mathbb{P}(Y \leq -y)}{\Phi(-y)} \right| \leq C_2 \frac{1 + y^3}{\Delta^{1/(1+2\gamma)}}. \quad (3.24)$$

Proof. The key observation is that $\psi(y)$ from (3.21) is bounded for $0 \leq y \leq q\Delta_\gamma$, where $q \in [0, 1]$ is fixed. Indeed, for such y , one can estimate $\psi(y) \leq c_1 + c_2 \Delta_\gamma^2 \exp(-c_3 \sqrt{\Delta_\gamma})$, where c_1 , c_2 and c_3 depend only on q . But the right-hand side of the last estimate can be bounded uniformly in Δ_γ .

Boundedness of ψ along with (3.20) and (3.22) implies that there exist universal constants D_1 , D_2 and D_3 , such that:

$$\exp\left(-\frac{D_2 y^3}{\Delta^{1/(1+2\gamma)}}\right) \left(1 - \frac{D_3(1+y)}{\Delta^{1/(1+2\gamma)}}\right) \leq \frac{\mathbb{P}(Y \geq y)}{1 - \Phi(y)} \leq \exp\left(\frac{D_2 y^3}{\Delta^{1/(1+2\gamma)}}\right) \left(1 + \frac{D_3(1+y)}{\Delta^{1/(1+2\gamma)}}\right) \quad (3.25)$$

for all $0 \leq y \leq D_1 \Delta^{1/(1+2\gamma)}$.

To derive (3.23), observe first that one can surely choose C_1 and C_2 , such that the desired estimate (3.23) holds in the region $\Delta^{1/(1+2\gamma)} \leq 3D_3$. However in the complementary region $\Delta^{1/(1+2\gamma)} > 3D_3$ and for $0 \leq y \leq \Delta^{1/(1+2\gamma)}/(3D_3)$, we have $D_3(1+y)\Delta^{-1/(1+2\gamma)} \leq 2/3$ and therefore:

$$\max \left\{ \left| \log \left(1 - \frac{D_3(1+y)}{\Delta^{1/(1+2\gamma)}} \right) \right|, \left| \log \left(1 + \frac{D_3(1+y)}{\Delta^{1/(1+2\gamma)}} \right) \right| \right\} \leq \frac{3 \log 3}{2} \frac{D_3(1+y)}{\Delta^{1/(1+2\gamma)}} \leq \frac{\log 3}{2} \frac{D_3(5+y^3)}{\Delta^{1/(1+2\gamma)}}. \quad (3.26)$$

The estimate (3.23) now follows from (3.25) and (3.26). Similarly, we obtain (3.24) and the proof is complete. \square

For the second part of Theorem 1.1, we shall need another result, which is due to Bentkus and Rudzkis [5] and appears as Lemma 2.4 on page 19 of Saulis and Statulevičius [27]. Like Lemma 3.4, we state it in a simplified form, which appears as a corollary of the afore-mentioned result.

Lemma 3.5. *Let Y be a random variable with $\mathbb{E}Y = 0$ and with its cumulants satisfying:*

$$|c^k[Y]| \leq \left(\frac{k!}{2}\right)^{1+\gamma} \frac{H}{\Delta^{k-2}}$$

for some $\gamma \geq 0$, $H > 0$ and $\Delta > 0$. Then for all $y \geq 0$, we have:

$$\mathbb{P}(Y \geq y) \leq \exp\left(-\frac{1}{4} \min\left\{\frac{y^2}{H}, (\Delta y)^{1/(1+\gamma)}\right\}\right).$$

Proof of Theorem 1.1.

(1): Applying Lemma 3.3 along with (1.2), we find that for λ large enough and $k \geq 2$, the cumulants of $\langle f, \bar{\mu}_{\lambda\kappa}^\xi \rangle$, i. e., the cumulants of $\langle f, \mu_{\lambda\kappa}^\xi \rangle$, i. e., $\langle f^k, c_\lambda^k \rangle$, satisfy:

$$\frac{|\langle f^k, c_\lambda^k \rangle|}{\sigma_\lambda^k[f]} \leq AB^k \|f\|_\infty^k (k!)^{d+2} \lambda^{(2-k)/2} \quad (3.27)$$

with some constants $A, B \geq 0$, where we recall from Subsection 1.3 that $\sigma_\lambda^2[f]$ denotes the variance of $\langle f, \bar{\mu}_{\lambda\kappa}^\xi \rangle$. To apply Corollary 3.1, rewrite the right-hand side of (3.27) as:

$$AB^2 \|f\|_\infty^2 (k!)^{d+2} \left(\frac{\sqrt{\lambda}}{B\|f\|_\infty}\right)^{-(k-2)} \leq (k!)^{d+2} \left(\frac{\sqrt{\lambda}}{B\|f\|_\infty \max\{1, AB^2\|f\|_\infty^2\}}\right)^{-(k-2)}.$$

Thus, recalling that the first cumulant of the centered measure $\bar{\mu}_{\lambda\kappa}^\xi$ is zero whereas its higher order cumulants coincide with those of $\mu_{\lambda\kappa}^\xi$, we can apply Corollary 3.1 to $Y := \langle f, \bar{\mu}_{\lambda\kappa}^\xi \rangle / \sigma_\lambda[f]$ with $\gamma = d+1$ and with Δ taken to be $\sqrt{\lambda}$ multiplied by some constant. It follows that there exist constants $D_1, D_2 \geq 0$, such that for all $0 \leq x \leq D_1 \sigma_\lambda[f] \lambda^{1/(6+4d)}$,

$$\left| \log \frac{\mathbb{P}(\langle f, \bar{\mu}_{\lambda\kappa}^\xi \rangle \geq x)}{1 - \Phi(x/\sigma_\lambda[f])} \right| \leq \frac{D_2}{\lambda^{1/(6+4d)}} \left[1 + \left(\frac{x}{\sigma_\lambda[f]} \right)^3 \right].$$

Applying (1.2) once again, we obtain that there exist constants $D_3, D_4 \geq 0$ such that for all $0 \leq x \leq D_3 \lambda^{(4+2d)/(6+4d)}$,

$$\left| \log \frac{\mathbb{P}(\langle f, \bar{\mu}_{\lambda\kappa}^\xi \rangle \geq x)}{1 - \Phi(x/\sigma_\lambda[f])} \right| \leq D_4 \left[\frac{1}{\lambda^{1/(6+4d)}} + \frac{x^3}{\lambda^{(10+6d)/(6+4d)}} \right].$$

An analogous bound holds for the lower tail probabilities and part (1) follows.

(2): From Lemma 3.3, we deduce after some calculation that the random variable $\langle f, \bar{\mu}_{\kappa\lambda}^\xi \rangle$ satisfies the conditions of Lemma 3.5 with:

$$H = \sigma_\lambda^{2t}[f] (D_5 \lambda)^{1-t} \quad \text{and} \quad \Delta = D_6 \min \left\{ 1, \frac{\sigma_\lambda^2[f]}{D_5 \lambda} \right\}^t$$

for all $0 \leq t \leq 1$, where D_5 and D_6 only depend on f, κ and ξ . We can suppose that $\sigma_\lambda^2[f] \leq D_5 \lambda$. By Lemma 3.5, we have:

$$\mathbb{P}(\pm \langle f, \bar{\mu}_{\lambda\kappa}^\xi \rangle \geq x) \leq \min_{0 \leq t \leq 1} \exp \left[-\frac{1}{4} \min \left\{ \frac{x^2}{\sigma_\lambda^{2t}[f] (D_5 \lambda)^{1-t}}, (D_6 x)^{1/(2+d)} \left(\frac{\sigma_\lambda^2[f]}{D_5 \lambda} \right)^{t/(2+d)} \right\} \right]. \quad (3.28)$$

An easy exercise in optimization shows that for all $a_1, a_2 > 0$, $b \geq 1$ and $c_1, c_2 \geq 0$, we have $\max_{0 \leq t \leq 1} \min \{a_1 b^{c_1 t}, a_2 b^{-c_2 t}\} = \min \{a_1 b^{c_1}, a_2, a_1^{c_2/(c_1+c_2)} a_2^{c_1/(c_1+c_2)}\}$. Plugging this into (3.28), we obtain (1.6). \square

3.4. Proof of moderate deviations

The results of Subsection 1.5 will follow from the following derivation of Theorem 1.1.

Lemma 3.6. *Let $\bar{\mu}_{\lambda\kappa}^\xi$ be defined as in Section 1 with ξ and κ satisfying Assumptions (*), let α_λ satisfy (1.8) and take $f \in \mathcal{B}(\mathbb{R}^d)$. Then, for $Q_\kappa^\xi(f) > 0$ and $t \geq 0$, we have:*

$$\lim_{\lambda \rightarrow \infty} \frac{1}{\alpha_\lambda^2} \log \mathbb{P}(\alpha_\lambda^{-1} \lambda^{-1/2} \langle f, \bar{\mu}_{\lambda\kappa}^\xi \rangle \geq t) = \lim_{\lambda \rightarrow \infty} \frac{1}{\alpha_\lambda^2} \log \mathbb{P}(\alpha_\lambda^{-1} \lambda^{-1/2} \langle f, \bar{\mu}_{\lambda\kappa}^\xi \rangle > t) = -\frac{t^2}{2Q_\kappa^\xi(f)}, \quad (3.29)$$

and for $Q_\kappa^\xi(f) = 0$ and $t > 0$, we have:

$$\lim_{\lambda \rightarrow \infty} \frac{1}{\alpha_\lambda^2} \log \mathbb{P}(\alpha_\lambda^{-1} \lambda^{-1/2} \langle f, \bar{\mu}_{\lambda\kappa}^\xi \rangle \geq t) = \lim_{\lambda \rightarrow \infty} \frac{1}{\alpha_\lambda^2} \log \mathbb{P}(\alpha_\lambda^{-1} \lambda^{-1/2} \langle f, \bar{\mu}_{\lambda\kappa}^\xi \rangle > t) = -\infty. \quad (3.30)$$

Proof. Suppose first that $Q_\kappa^\xi(f) > 0$. In this case, we plug $x = t \alpha_\lambda \lambda^{1/2}$ into (1.4) and make use of the fact that for all $y \geq 0$,

$$\frac{1}{2 + \sqrt{2\pi} y} \leq e^{y^2/2} (1 - \Phi(y)) \leq \frac{1}{2}.$$

Combining both, we obtain the bound:

$$\left| \log \mathbb{P}(\alpha_\lambda^{-1} \lambda^{-1/2} \langle f, \bar{\mu}_{\lambda\kappa}^\xi \rangle \geq t) + \frac{\alpha_\lambda^2 \lambda t^2}{2\sigma_\lambda^2[f]} \right| \leq \log \left(2 + \frac{\sqrt{2\pi} t_0 \alpha_\lambda \lambda^{1/2}}{\sigma_\lambda[f]} \right) + C_2 \frac{1 + \alpha_\lambda^3 t_0^3}{\lambda^{1/(6+4d)}}.$$

Dividing by α_λ^2 , making use of (1.2) and applying condition (1.8), this implies (3.29) for ‘greater or equal’. The corresponding result for the strict inequality follows by continuity.

In the case where $Q_\kappa^\xi(f) = 0$, plug $x = t \alpha_\lambda \lambda^{1/2}$ into (1.6) to obtain:

$$\frac{1}{\alpha_\lambda^2} \log \mathbb{P}(\alpha_\lambda^{-1} \lambda^{-1/2} \langle f, \bar{\mu}_{\lambda\kappa}^\xi \rangle \geq t) \leq -t^2 \min \left\{ C_4 \frac{\lambda}{\sigma_\lambda^2[f]}, C_5 \left(\frac{\lambda}{(t \alpha_\lambda)^{6+4d}} \right)^{\frac{1}{4+2d}}, C_6 \left(\frac{\lambda}{(t \alpha_\lambda)^{6+4d}} \right)^{\frac{1}{6+2d}} \right\}$$

and the desired limiting behavior follows again from (1.2) and (1.8). This completes the proof. \square

Proof of Theorem 1.2. We apply the preceding lemma along with Theorem 4.1.11 in [10], which allows us to derive a LDP from the limiting behavior of probabilities for a basis of topology. For the latter, we choose all open intervals (a, b) , where at least one of the endpoints is finite and where none of the endpoints is zero. Denote the family of all such intervals by \mathcal{U} . From Lemma 3.6, it follows that for each $U = (a, b) \in \mathcal{U}$,

$$\mathcal{L}_U := - \lim_{\lambda \rightarrow \infty} \frac{1}{\alpha_\lambda^2} \log \mathbb{P}(\alpha_\lambda^{-1} \lambda^{-1/2} \langle f, \bar{\mu}_{\lambda\kappa}^\xi \rangle \in U) = \begin{cases} b^2/(2Q_\kappa^\xi(f)) & ; \quad a < b < 0 \\ 0 & ; \quad a < 0 < b \\ a^2/(2Q_\kappa^\xi(f)) & ; \quad 0 < a < b \end{cases},$$

for all $U \in \mathcal{U}$, recalling our convention on division by zero from the end of Subsection 1.2. By Theorem 4.1.11 in [10], the random variables $\alpha_\lambda^{-1} \lambda^{-1/2} \langle f, \bar{\mu}_{\lambda\kappa}^\xi \rangle$ satisfy a *weak* LDP (MDP) as $\lambda \rightarrow \infty$ with speed α_λ^2 and rate function:

$$t \mapsto \sup_{\substack{U \in \mathcal{U} \\ t \in U}} \mathcal{L}_U = \frac{t^2}{2Q_\kappa^\xi(f)},$$

which matches the function $I_{\kappa;f}^\xi$ from (1.9). Here, weak LDP means that the lower bound in (1.7) holds for all measurable sets Γ , while the upper bound holds for all relatively compact measurable Γ . However, from Lemma 3.6, it follows that the family $\alpha_\lambda^{-1} \lambda^{-1/2} \langle f, \bar{\mu}_{\lambda\kappa}^\xi \rangle$ is exponentially tight for speed α_λ^2 , i. e., for each $M < \infty$, there exists a measurable relatively compact set K , such that $\limsup_{\lambda \rightarrow \infty} \alpha_\lambda^{-2} \log \mathbb{P}(\alpha_\lambda^{-1} \lambda^{-1/2} \langle f, \bar{\mu}_{\lambda\kappa}^\xi \rangle \notin K) \leq -M$. By Lemma 1.2.18 in [10], the family $\alpha_\lambda^{-1} \lambda^{-1/2} \langle f, \bar{\mu}_{\lambda\kappa}^\xi \rangle$ must then satisfy a *full* LDP with the same speed and the same *good* rate function. This completes the proof. \square

Now we turn to the proof of Theorem 1.3. We begin with the same argument, but with a more sophisticated choice of the basis of topology.

Lemma 3.7. *Let F and V be pairwise dual finite-dimensional vector spaces, equipped with the usual topology, and let Q be a positively semi-definite quadratic form on F . Let \mathcal{U}_0 be the family of all open subsets of the half-spaces $\{\nu \in V ; \langle f, \nu \rangle > a\}$, where $a > 0$ and $Q(f) = 0$, and denote by \mathcal{U}_1 the family of all sets of the form*

$$\{\nu \in V ; \langle f_0, \nu \rangle > a_0, \langle f_1, \nu \rangle < a_1, \langle f_2, \nu \rangle < a_2, \dots, \langle f_n, \nu \rangle < a_n\}, \quad (3.31)$$

where either $0 < a_0/\sqrt{Q(f_0)} < a_i/\sqrt{Q(f_i)}$ for all $i = 1, 2, \dots, n$, or $a_0 < 0 < a_i$ for all $i = 1, 2, \dots, n$. Then the family $\mathcal{U}_0 \cup \mathcal{U}_1$ is a basis of the topology on V .

Proof. Define $F_0 := \{f \in F ; Q(f) = 0\}$ and $F_0^\perp := \{\nu \in V ; \langle f, \nu \rangle = 0 \text{ for all } f \in F_0\}$. Now take $\mu \in V$ and its open neighborhood W . We have to show that there exists a $U \in \mathcal{U}_0 \cup \mathcal{U}_1$, such that $x \in U \subseteq W$. We distinguish three cases.

Case 1: $\mu \notin F_0^\perp$. Then there exists an $f \in F_0$ and an $a > 0$, such that $\langle f, \mu \rangle > a$, so that we can take $U := W \cap \{\nu \in V ; \langle f, \mu \rangle > a\} \in \mathcal{U}_0$.

Case 2: $\mu = 0$. Then there exists an $\varepsilon > 0$ and elements f_0, f_1, \dots, f_n , such that $U := \bigcap_{i=0}^n \{\nu ; \langle f_i, \nu \rangle < \varepsilon\} \subseteq W$. Clearly, $0 \in U$. Since we can write $U = \{\nu ; \langle -f_0, \nu \rangle > -\varepsilon\} \cap \bigcap_{i=1}^n \{\nu ; \langle f_i, \nu \rangle < \varepsilon\}$, we also have $U \in \mathcal{U}_1$.

Case 3: $\mu \in F_0^\perp \setminus \{0\}$. Recalling our convention on division by zero from the end of Subsection 1.2, it follows from standard linear algebra and topology that the map $f \mapsto \langle f, \mu \rangle / \sqrt{Q(f)}$ vanishes on F_0 , is continuous on $F \setminus \{0\}$, is bounded and that attains its maximum, say, at f_0 . Since $\mu \neq 0$, we also have $\langle f_0, \mu \rangle > 0$ and $Q(f_0) > 0$.

There exist functions f_1, f_2, \dots, f_n and a $\delta \in (0, \langle f_0, \mu \rangle)$, such that $U_0 := \{\nu ; \langle f_0, \nu - \mu \rangle > -\delta\} \cap \bigcap_{i=1}^n \{\nu ; \langle f_i, \nu - \mu \rangle < \delta\} \subseteq W$. Now consider the sets:

$$U_{\varepsilon, t} := \{\nu \in V ; \langle f_0, \nu - \mu \rangle > -\varepsilon\} \cap \bigcap_{i=1}^n \{\langle f_i + tf_i, \nu - \mu \rangle < \varepsilon\}.$$

Clearly, $\mu \in U_{\varepsilon, t}$ for all $\varepsilon, t > 0$. Next, for each $\nu \in U_{\varepsilon, t}$, we can estimate:

$$\langle f_i, \nu - \mu \rangle = \frac{1}{t} [\langle f_0 + tf_i, \nu - \mu \rangle - \langle f_0, \nu - \mu \rangle] < \frac{2\varepsilon}{t}.$$

Therefore, if $\varepsilon < \delta$ and $2\varepsilon/t < \delta$, then $U_{\varepsilon, t} \subseteq U_0$.

Now we turn our attention to the question when $U_{\varepsilon, t} \in \mathcal{U}$. By (3.31), this will be surely true if:

$$\frac{\langle f_0 + tf_i, \mu \rangle + \varepsilon}{\sqrt{Q(f_0 + tf_i)}} > \frac{\langle f_0, \mu \rangle - \varepsilon}{\sqrt{Q(f_0)}}. \quad (3.32)$$

Since $\langle f, \mu \rangle / \sqrt{Q(f)}$ is maximal at f_0 , it follows from smoothness that there exist $a, b > 0$, such that for all $i = 1, 2, \dots, n$,

$$\frac{\langle f_0 + tf_i, \mu \rangle}{\sqrt{Q(f_0 + tf_i)}} \geq \frac{\langle f_0, \mu \rangle}{\sqrt{Q(f_0)}} - at^2$$

for all $0 \leq t \leq b$. Therefore, the condition (3.32) is satisfied if $t \leq b$ and $at^2 < \varepsilon / \sqrt{Q(f_0)}$. Collecting everything together, we conclude after some calculation that if $0 < t < \min \left\{ b, \frac{\delta}{2a\sqrt{Q(f)}}, \sqrt{\frac{\delta}{a\sqrt{Q(f)}}} \right\}$, then there exists an $\varepsilon > 0$, such that $U_{\varepsilon, t} \subseteq U_0 \subseteq W$ and $U_{\varepsilon, t} \in \mathcal{U}$, so that the desired set U exists in this case, too. This completes the proof. \square

Proof of Theorem 1.3. We split the argument into several steps.

Step 1: derive a MDP for finite-dimensional restrictions of the measures. Denote by \mathcal{F} the set of all finite-dimensional subspaces of $\mathcal{B}(\mathbb{R}^d)$, fix $F \in \mathcal{F}$ and consider the random measures $\mu_{\lambda\kappa}^\xi$ as linear functionals on F . Let $\mathcal{U}_0 \cup \mathcal{U}_1$ be the basis of the usual topology on F' , where \mathcal{U}_0 and \mathcal{U}_1 are defined as in Lemma 3.7, taking the quadratic form Q_κ^ξ . For $U = \{\nu \in F' ; \langle f_0, \nu \rangle > a_0, \langle f_1, \nu \rangle < a_1, \dots, \langle f_n, \nu \rangle < a_n\} \in \mathcal{U}_1$, where the numbers a_0, a_1, \dots, a_n satisfy the conditions below (3.31), we have, by Lemma 3.6:

$$\mathcal{L}_U := - \lim_{\lambda \rightarrow \infty} \alpha_\lambda^{-2} \mathbb{P} \left(\alpha_\lambda^{-1} \lambda^{-1/2} \bar{\mu}_{\lambda\kappa}^\xi \in U \right) = \frac{(\max\{a_0, 0\})^2}{2Q_\kappa^\xi(f_0)};$$

for $U \in \mathcal{U}_0$, we have $\mathcal{L}_U = \infty$. By Theorem 4.1.11 in [10], the random functionals $\alpha_\lambda^{-1} \lambda^{-1/2} \bar{\mu}_{\lambda\kappa}^\xi \in F'$ satisfy a weak LDP (MDP) as $\lambda \rightarrow \infty$ with speed α_λ^2 and rate function:

$$\nu \mapsto \sup_{\substack{U \in \mathcal{U} \\ \nu \in U}} \mathcal{L}_U = \sup_{f \in F} \frac{\langle f, \nu \rangle^2}{2Q_\kappa^\xi(f)}.$$

Finally, by Lemma 1.2.18 in [10], these random functionals satisfy a full LDP (MDP) with the same good rate function because they are exponentially tight for speed α_λ^2 . To see this, take a basis f_1, \dots, f_n of F with $Q_\kappa^\xi(f_i) \leq 1$ for all i and consider the compact sets $K_M := \bigcap_{i=1}^n \{\nu \in F' ; |\langle f_i, \nu \rangle| \leq M\}$. By Lemma 3.6, $\limsup_{\lambda \rightarrow \infty} \alpha_\lambda^{-2} \mathbb{P}(\alpha_\lambda^{-1} \lambda^{-1/2} \bar{\mu}_{\lambda\kappa}^\xi \notin K_M) \leq -M^2/2$ and exponential tightness follows. This completes Step 1.

Step 2: combine the MDP's for finite-dimensional restrictions into a MDP for entire random measures. We apply a version of the Dawson–Gärtner theorem for projective limits, namely Theorem 4.6.9 in [10], naturally embedding $\mathcal{M}(\mathbb{R}^d)$ into $(\mathcal{B}(\mathbb{R}^d))'$, the algebraic dual of $\mathcal{B}(\mathbb{R}^d)$, and identifying the projections of the functionals to finite-dimensional spaces with their restrictions to finite-dimensional subspaces of $\mathcal{B}(\mathbb{R}^d)$. Thus we find that, as $\lambda \rightarrow \infty$, the random measures $\alpha_\lambda^{-1} \lambda^{-1/2} \bar{\mu}_{\lambda\kappa}^\xi$ satisfy the MDP in $(\mathcal{B}(\mathbb{R}^d))'$ with speed α_λ^2 and the good rate function:

$$J_\kappa^\xi(\nu) := \sup_{F \in \mathcal{F}} \sup_{f \in \text{Lin } F} \frac{\langle f, \nu \rangle^2}{2Q_\kappa^\xi(f)} = \sup_{f \in \mathcal{B}(\mathbb{R}^d)} \frac{\langle f, \nu \rangle^2}{2Q_\kappa^\xi(f)}.$$

Step 3: compute the rate function. Take $\nu \in (\mathcal{B}(\mathbb{R}^d))'$ and distinguish five separate cases.

Case 1: ν is unbounded with respect to the supremum norm on $\mathcal{B}(\mathbb{R}^d)$. Since the latter is stronger than the seminorm $\sqrt{Q_\kappa^\xi}$, we have $J_\kappa^\xi(\nu) = +\infty$ in this case.

Case 2: ν is bounded with respect to the supremum norm, but is not a measure. This means that there exists a sequence of bounded functions f_n with $f_n \downarrow 0$ pointwise, such that the $\langle f_n, \nu \rangle$ does not converge to 0. We may assume that $|\langle f_n, \nu \rangle| \geq 1$ for all n . Denoting $L^2 := \{f: \mathbb{R}^d \rightarrow \mathbb{R} ; Q_\kappa^\xi(f) < \infty\}$ and noting that $\mathcal{B}(\mathbb{R}^d) \subseteq L^2$, we find that $Q_\kappa^\xi(f) \rightarrow 0$ by the dominated convergence theorem. Therefore $J_\kappa^\xi(\nu) = +\infty$.

Case 3: ν is a measure, but is not absolutely continuous with respect to $V^\xi(\kappa(x)) \kappa(x) dx$. In this case, there exists a measurable set A with $\int_A V^\xi(\kappa(x)) \kappa(x) dx = 0$, but $\nu(A) \neq 0$. In other words, $Q_\kappa^\xi(\mathbf{1}_A) = 0$, but $\langle \mathbf{1}_A, \nu \rangle \neq 0$, so that again $J_\kappa^\xi(\nu) = +\infty$.

Case 4: $\nu \ll V^\xi(\kappa(x)) \kappa(x) dx$, but $Q_\kappa^\xi(\rho) = \infty$, where $\rho(x) := \nu(dx)/(V^\xi(\kappa(x)) \kappa(x) dx)$. In this case, there exists a sequence $\rho_1, \rho_2, \dots \in L^2$ which converges pointwise to ρ and satisfies $\rho \rho_n \geq 0$ and $|\rho_1| \leq |\rho_2| \leq \dots$. By the monotone convergence theorem, we have $Q_\kappa^\xi(\rho_n) \uparrow \infty$; we may assume that $Q_\kappa^\xi(\rho_1) > 0$. Then the functions $g_n := \rho_n / Q_\kappa^\xi(\rho_n)$ satisfy $Q_\kappa^\xi(g_n) = 1/Q_\kappa^\xi(\rho_n) \rightarrow 0$ but $\langle g_n, \nu \rangle \geq 1$, so that again $J_\kappa^\xi(\nu) = +\infty$.

Case 5: $\nu \ll V^\xi(\kappa(x)) \kappa(x) \, dx$ and $\rho(x) := \nu(dx)/(V^\xi(\kappa(x)) \kappa(x) \, dx)$ satisfies $Q_\kappa^\xi(\rho) < \infty$. In this case we may write:

$$\begin{aligned} J_\kappa^\xi(\nu) &= \sup_{f \in \mathcal{B}(\mathbb{R}^d)} \frac{\left(\int_{\mathbb{R}^d} f(x) \rho(x) V^\xi(\kappa(x)) \kappa(x) \, dx \right)^2}{2 \int_{\mathbb{R}^d} \rho^2(x) V^\xi(\kappa(x)) \kappa(x) \, dx} = \sup_{f \in L^2} \frac{\left(\int_{\mathbb{R}^d} f(x) \rho(x) V^\xi(\kappa(x)) \kappa(x) \, dx \right)^2}{2 \int_{\mathbb{R}^d} \rho^2(x) V^\xi(\kappa(x)) \kappa(x) \, dx} = \\ &= \frac{1}{2} Q_\kappa^\xi(\rho) = I_\kappa^\xi(\nu), \end{aligned}$$

where the latter is defined in (1.10). The second equality holds because $\mathcal{B}(\mathbb{R}^d)$ is dense in L^2 ; the third one is due to the Cauchy–Schwarz inequality. This completes Step 3.

Step 4: restrict the MDP. To see that we may replace $(\mathcal{B}(\mathbb{R}^d))'$ and J_κ^ξ with $\mathcal{M}(\mathbb{R}^d)$ and I_κ^ξ , we apply Lemma 4.1.5 in [10], noting that J_κ^ξ agrees with I_κ^ξ on $\mathcal{M}(\mathbb{R}^d)$ and is infinite outside $\mathcal{M}(\mathbb{R}^d)$. This completes the proof. \square

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